

COEFFICIENT INEQUALITY MAKING RESULTS SHARP FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

Shekhar Choudhury¹

¹Assistant Professor, Department Of Mathematics, Udaguri College, Udaguri, Assam- 784509, India.

DOI: <https://www.doi.org/10.58257/IJPREMS32481>

ABSTRACT

We have introduced subclasses of analytic functions and have obtained sharp upper bounds of the Fekete Szego functional $|a_3 - \mu a_2^2|$ for the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ belonging to these classes and subclasses.

Keywords: Univalent functions, Starlike functions, Close to convex functions and bounded functions.

1. INTRODUCTION

Let \mathcal{A} denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

analytic in the unit disc given by $\mathbb{E} = \{z: |z| < 1\}$. Let \mathcal{S} be the class of analytic functions of the form (1.1), which are univalent in \mathbb{E} . In 1916, Bieber Bach ([1], [2]) proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner [10] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$.

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and a_2^2 for the class \mathcal{S} , Fekete and Szegö [4] used Löwner's method to prove the following well known result for the class \mathcal{S} .

Let $f(z) \in \mathcal{S}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq 0; \\ 1 + 2 \exp\left(\frac{-2\mu}{1-\mu}\right), & \text{if } 0 \leq \mu \leq 1; \\ 4\mu - 3, & \text{if } \mu \geq 1. \end{cases} \quad (1.2)$$

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes \mathcal{S} ([3], [9]).

Let us define some subclasses of \mathcal{S} .

We denote by \mathcal{S}^* , the class of univalent starlike functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A} \text{ and satisfying the condition} \\ \operatorname{Re} \left(\frac{zg'(z)}{g(z)} \right) > 0, z \in \mathbb{E}. \quad (1.3)$$

We denote by \mathcal{K} , the class of univalent convex functions and satisfying the condition

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in \mathcal{A} \\ \operatorname{Re} \frac{(zh'(z))'}{h'(z)} > 0, z \in \mathbb{E}. \quad (1.4)$$

A function $f(z) \in \mathcal{A}$ is said to be close to convex if there exists $g(z) \in \mathcal{S}^*$ such that

$$\operatorname{Re} \left(\frac{zf'(z)}{g(z)} \right) > 0, z \in \mathbb{E}. \quad (1.5)$$

The class of close to convex functions is denoted by \mathcal{C} and was introduced by Kaplan [7] and it was shown by him that all close to convex functions are univalent.

$$\mathcal{S}^*(A, B) = \left\{ f(z) \in \mathcal{A}; \frac{zf'(z)}{f(z)} < \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E} \right\} \quad (1.6)$$

$$\mathcal{K}(A, B) = \left\{ f(z) \in \mathcal{A}; \frac{(zf'(z))'}{f'(z)} < \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E} \right\} \quad (1.7)$$

It is obvious that $\mathcal{S}^*(A, B)$ is a subclass of \mathcal{S}^* and $\mathcal{K}(A, B)$ is a subclass of \mathcal{K} .

Several authors studied and introduced various classes and subclasses of univalent analytic functions and established Fekete Szego inequality for the same. ([3]-[9], [12]-[15], [22]-[62])

N. Kaur [11] introduced a new subclass as and have established its coefficient inequality.

$$S^*(f, f', \alpha, \beta) = \left\{ f(z) \in \mathcal{A}; (1 - \alpha) \left(\frac{zf'(z)}{f(z)} \right)^\beta + \alpha \left(\frac{(zf'(z))'}{f'(z)} \right)^{1-\beta} \prec \frac{1+z}{1-z}; z \in \mathbb{E} \right\}$$

We will deal with the subclass of $S^*(f, f', \alpha, \beta)$ defined as follows in the present paper:

$$S^*(f, f', \alpha, \beta, A, B) = \left\{ f(z) \in \mathcal{A}; (1 - \alpha) \left(\frac{zf'(z)}{f(z)} \right)^\beta + \alpha \left(\frac{(zf'(z))'}{f'(z)} \right)^{1-\beta} \prec \frac{1+Az}{1+Bz}; z \in \mathbb{E} \right\} \quad (1.8)$$

We will deal with the subclass $S^*(f, f', \alpha, \beta, \delta)$ defined as follows in our next paper:

$$S^*(f, f', \alpha, \beta, \delta) = \left\{ f(z) \in \mathcal{A}; (1 - \alpha) \left(\frac{zf'(z)}{f(z)} \right)^\beta + \alpha \left(\frac{(zf'(z))'}{f'(z)} \right)^{1-\beta} \prec \left(\frac{1+z}{1-z} \right)^\delta; z \in \mathbb{E} \right\} \quad (1.9)$$

Symbol \prec stands for subordination, which we define as follows:

Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in \mathbb{E} . Then $f(z)$ is called subordinate to $F(z)$ in \mathbb{E} if there exists a function $w(z)$ analytic in \mathbb{E} satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$; $z \in \mathbb{E}$ and we write $f(z) \prec F(z)$.

By \mathcal{U} , we denote the class of analytic bounded functions of the form $w(z) = \sum_{n=1}^{\infty} d_n z^n$, $w(0) = 0$, $|w(z)| < 1$.
(1.10)

It is known that $|d_1| \leq 1$, $|d_2| \leq 1 - |d_1|^2$.
(1.11)

PRELIMINARY LEMMAS:

For $0 < c < 1$, we write $w(z) = \left(\frac{c+z}{1+cz} \right)$ so that

$$\frac{1+w(z)}{1-w(z)} = 1 + 2cz + 2z^2 + \dots \quad (2.1)$$

2. MAIN RESULTS

THEOREM 2.1

Let $f(z) \in S^*(f, f', \alpha, \beta, A, B)$, then The results are sharp.

$$\left\{ \begin{array}{l} |a_3 - \mu a_2^2| \\ \frac{(A-B)^2(8\alpha + 3\beta + 4\alpha^2 - 12\alpha^2\beta - 9\alpha\beta^2 - 7\alpha\beta)}{4(3\alpha + \beta - 4\alpha\beta)\{(1-\alpha)\beta + 2\alpha(1-\beta)\}^2} - \frac{(A-B)^2}{\{(1-\alpha)\beta + 2\alpha(1-\beta)\}^2} \mu, \\ \text{if } \mu \leq \frac{(A-B)8\alpha + 3\beta + 4\alpha^2 - 12\alpha^2\beta - 9\alpha\beta^2 - 7\alpha\beta - 4\{(1-\alpha)\beta + 2\alpha(1-\beta)\}^2}{(3\alpha + \beta - 4\alpha\beta)}; \end{array} \right. \quad (3.1)$$

$$\leq \left\{ \begin{array}{l} \frac{(A-B)}{2(3\alpha + \beta - 4\alpha\beta)} \\ \text{if } \frac{(A-B)8\alpha + 3\beta + 4\alpha^2 - 12\alpha^2\beta - 9\alpha\beta^2 - 7\alpha\beta - 4\{(1-\alpha)\beta + 2\alpha(1-\beta)\}^2}{(3\alpha + \beta - 4\alpha\beta)} \leq \mu \leq \\ \frac{4\{(1-\alpha)\beta + 2\alpha(1-\beta)\}^2 - (A-B)8\alpha + 3\beta + 4\alpha^2 - 12\alpha^2\beta - 9\alpha\beta^2 - 7\alpha\beta}{(3\alpha + \beta - 4\alpha\beta)}; \end{array} \right. \quad (3.2)$$

$$\left. \begin{array}{l} \frac{(A-B)^2}{\{(1-\alpha)\beta + 2\alpha(1-\beta)\}^2} \mu - \frac{(A-B)^2(8\alpha + 3\beta + 4\alpha^2 - 12\alpha^2\beta - 9\alpha\beta^2 - 7\alpha\beta)}{4(3\alpha + \beta - 4\alpha\beta)\{(1-\alpha)\beta + 2\alpha(1-\beta)\}^2}, \\ \text{if } \mu \geq \frac{4\{(1-\alpha)\beta + 2\alpha(1-\beta)\}^2 - (A-B)8\alpha + 3\beta + 4\alpha^2 - 12\alpha^2\beta - 9\alpha\beta^2 - 7\alpha\beta}{(3\alpha + \beta - 4\alpha\beta)} \end{array} \right. \quad (3.3)$$

Proof 2.2. By definition of $S^*(f, f', \alpha, \beta, A, B)$, we have

$$(1 - \alpha) \left(\frac{zf'(z)}{f(z)} \right)^\beta + \alpha \left(\frac{(zf'(z))'}{f'(z)} \right)^{1-\beta} = \frac{1+Aw(z)}{1+Bw(z)}; w(z) \in \mathcal{U}. \quad (3.4)$$

Expanding the series (3.4), we get

$$(1 - \alpha) \left\{ 1 + \beta a_2 z + (2\beta a_3 + \frac{\beta(\beta-3)}{2} a_2^2) z^2 + \dots \right\} + \alpha \{ 1 + 2(1-\beta) a_2 z + 2(1-\beta)(3a_3 - (\beta + 2)a_2^2) z^2 + \dots \} = (1 + (A-B)c_1 z + (A-B)(c_2 - Bc_1^2) z^2 + \dots). \quad (3.5)$$

Identifying terms in (3.5), we get

$$a_2 = \frac{(A-B)}{(1-\alpha)\beta+2\alpha(1-\beta)} c_1 \quad (3.6)$$

$$a_3 = \frac{(A-B)}{2(3\alpha+\beta-4\alpha\beta)} c_2 + \frac{(A-B)^2(8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta)}{4(3\alpha+\beta-4\alpha\beta)\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} c_1^2. \quad (3.7)$$

From (3.6) and (3.7), we obtain

$$a_3 - \mu a_2^2 = \frac{(A-B)}{2(3\alpha+\beta-4\alpha\beta)} c_2 + \left[\frac{(A-B)^2(8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta)}{4(3\alpha+\beta-4\alpha\beta)\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} - \frac{(A-B)^2}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} \mu \right] c_1^2. \quad (3.8)$$

Taking absolute value, (3.8) can be rewritten as

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{3\alpha+\beta-4\alpha\beta} |c_2| + \frac{(A-B)^2}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} \left| \frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)} - \mu \right| |c_1^2|. \quad (3.9)$$

Using (1.9) in (3.9), we get

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{(A-B)}{3\alpha+\beta-4\alpha\beta} (1 - |c_1|^2) \\ &\quad + \frac{(A-B)^2}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} \left| \frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)} - \mu \right| |c_1^2| \\ &= \frac{(A-B)}{3\alpha+\beta-4\alpha\beta} + \frac{(A-B)^2}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} \left[\left| \frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)} - \mu \right| - \frac{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{(A-B)(3\alpha+\beta-4\alpha\beta)} \right] |c_1|^2. \end{aligned} \quad (3.10)$$

Case I: $\mu \leq \frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)}$.

(3.10) can be rewritten as

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{3\alpha+\beta-4\alpha\beta} + \frac{(A-B)^2}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} \left[\frac{(A-B)8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta-4\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{(3\alpha+\beta-4\alpha\beta)} - \mu \right] |c_1|^2. \quad (3.11)$$

Subcase I (a): $\mu \leq \frac{(A-B)8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta-4\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{(3\alpha+\beta-4\alpha\beta)}$.

Using (1.9), (3.11) becomes

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)^2(8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta)}{4(3\alpha+\beta-4\alpha\beta)\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} - \frac{(A-B)^2}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} \mu \quad (3.12)$$

$$\text{Subcase I (b): } \mu \geq \frac{(A-B)8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta-4\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{(3\alpha+\beta-4\alpha\beta)}.$$

We obtain from (3.11)

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{3\alpha+\beta-4\alpha\beta}. \quad (3.13)$$

$$\text{Case II: } \mu \geq \frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)}$$

Preceding as in case I, we get

$$|a_3 - \mu a_2^2| \leq \frac{1}{3\alpha+\beta-4\alpha\beta} + \frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} \left[\mu - \frac{4\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2 - (A-B)8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)} \right] |c_1|^2. \quad (3.14)$$

$$\text{Subcase II (a): } \mu \leq \frac{4\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2 - (A-B)8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)}$$

(3.14) takes the form

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{3\alpha+\beta-4\alpha\beta} \quad (3.15)$$

Combining subcase I (b) and subcase II (a), we obtain

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{(A-B)}{3\alpha+\beta-4\alpha\beta} \text{ if } \frac{(A-B)8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta-4\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{(3\alpha+\beta-4\alpha\beta)} \leq \mu \leq \\ &\quad \frac{4\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2 - (A-B)8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)} \end{aligned} \quad (3.16)$$

$$\text{Subcase II (b): } \mu \geq \frac{4\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2 - (A-B)8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)}$$

Preceding as in subcase I (a), we get

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)^2}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} \mu - \frac{(A-B)^2(8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta)}{4(3\alpha+\beta-4\alpha\beta)\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} \quad (3.17)$$

Combining (3.12), (3.16) and (3.17), the theorem is proved.

Corollary 2.3: Putting $\alpha = 1, \beta = 0, A = 1, B = -1$ in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} 1 - \mu, & \text{if } \mu \leq 1; \\ \frac{1}{3}, & \text{if } 1 \leq \mu \leq \frac{4}{3}; \\ \mu - 1, & \text{if } \mu \geq \frac{4}{3} \end{cases}$$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent convex functions.

Corollary 2.3: Putting $\alpha = 0, \beta = 1, A = 1, B = -1$ in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq \frac{1}{2}; \\ 1, & \text{if } \frac{1}{2} \leq \mu \leq 1; \\ 4\mu - 3, & \text{if } \mu \geq 1 \end{cases}$$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent starlike functions.

Corollary 2.4: Putting $A = 1, B = -1$ in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{\{(1-\alpha)\beta + 2\alpha(1-\beta)\}^2} \left[\frac{8\alpha + 3\beta + 4\alpha^2 - 12\alpha^2\beta - 9\alpha\beta^2 - 7\alpha\beta}{(3\alpha + \beta - 4\alpha\beta)} - 4\mu \right], \\ \text{if } \mu \leq \frac{8\alpha + 3\beta + 4\alpha^2 - \beta^2 - 3\alpha\beta^2 - 7\alpha\beta}{4(3\alpha + \beta - 4\alpha\beta)}; \\ \frac{1}{3\alpha + \beta - 4\alpha\beta} \\ \text{if } \frac{8\alpha + 3\beta + 4\alpha^2 - \beta^2 - 3\alpha\beta^2 - 7\alpha\beta}{4(3\alpha + \beta - 4\alpha\beta)} \leq \mu \leq \\ \frac{8\alpha + 3\beta + 8\alpha^2 + \beta^2 - 24\alpha^2\beta - 6\alpha\beta^2 - 7\alpha\beta}{4(3\alpha + \beta - 4\alpha\beta)}; \\ \frac{1}{\{(1-\alpha)\beta + 2\alpha(1-\beta)\}^2} \left[4\mu - \frac{8\alpha + 3\beta + 4\alpha^2 - 12\alpha^2\beta - 9\alpha\beta^2 - 7\alpha\beta}{(3\alpha + \beta - 4\alpha\beta)} \right], \\ \text{if } \mu \geq \frac{8\alpha + 3\beta + 8\alpha^2 + \beta^2 - 24\alpha^2\beta - 6\alpha\beta^2 - 7\alpha\beta}{4(3\alpha + \beta - 4\alpha\beta)} \end{cases}$$

These estimates were derived by N. Kaur [11] and are results for the subclass $S^*(f, f', \alpha, \beta)$ of univalent starlike functions.

3. REFERENCES

- [1] Alexander, J.W Function which map the interior of unitcircle upon simple regions, Ann. Of Math., 17 (1995), 12-22.
- [2] Aoufet. al.,Fekete – Szego Inequalities for p – valentstarlike and convex functions of complex order, Journal of the Egyptian Mathematical Society,22 (2014), 190 – 196.
- [3] Bieberbach, L. Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskrises vermitteln,S. – B. Preuss. Akad. Wiss. 38 (1916), 940-955.
- [4] De Branges L., A proof of Bieberbach Conjecture, Acta. Math., 154 (1985), 137-152.
- [5] Duren, P.L., Coefficient of univalent functions, Bull. Amer. Math. Soc., 83 (1977), 891-911.
- [6] Fekete, M. and Szegö, G, Eine Bemerkung über ungerade schlichte Funktionen, J.London Math. So 8 (1933) 85-89.
- [7] Garabedian, P.R., Schiffer, M., A Proof for the Bieberbach Conjecture for the fourth coefficient, Arch. Rational Mech. Anal., 4 (1955), 427-465.
- [8] Kaur, C. and Singh, G., Approach To Coefficient Inequality For A New Subclass Of Starlike Functions With Extremals, International Journal Of Research In Advent Technology, 5(2017)
- [9] Kaur, C. and Singh, G., Coefficient Problem For A New Subclass Of Analytic Functions Using Subordination, International Journal Of Research In Advent Technology, 5(2017)
- [10] Kaur. G, Singh. G, Arif. M, Chinram. R, Iqbal. J, A study of third and fourth Hankel determinant problem for a particular class of bounded turning functions, Mathematical Problems in Engineering, 22, 511-526, 2021
- [11] Kaur N., Fekete Szego Inequality Alongwith Their Extremal Functions Making Results Sharp For Certain Subclasses Of Analytic Functions, Webology, 18(4), 2021, 3075-3083

[12] Keogh, F.R., Merkes, E.P., A coefficient inequality for certain classes of analytic functions, Proc. Of Amer. Math. Soc., 20, 8-12, 1989.

[13] Koebe, P., Über Die uniformisierung beliebiger analytischer Kurven, Nach. Ges. Wiss. Gottingen (1907), 633-669.

[14] Lindelof, E., Memoire sur certaines inégalités dans la théorie des fonctions monogènes et sur quelques propriétés nouvelles de ces fonctions dans la voisinage d'un point singulier essentiel, Acta Soc. Sci. Fenn., 23 (1909), 481-519.

[15] Löewner, C. (1917), Untersuchungen über die Verzerrung bei Konformen Abbildung des Einheitskreises $|z| < 1$, die durch Funktionen mit nicht verschwindender Ableitung geliefert werden, Ber. Verh. Sächs. Ges. Wiss. Leipzig, Vol. 69, pp.89-106

[16] Ma, W. and Minda, D. unified treatment of some special classes of univalent functions, In Proceedings of the Conference on Complex Analysis , Int. Press Tianjin (1994), 157-169.

[17] Miller, S.S., Mocanu, P.T. And Reade, M.O., All convex functions are univalent and starlike, Proc. of Amer. Math. Soc., 37 (1973), 553-554.

[18] Nehari, Z. (1952), Conformal Mappings, McGraw- Hill, New York.

[19] Nevanlinna, R., Über die Eigenschaft einer analytischen Funktion in der Umgebung eines singulären Punktes, Acta Soc. Sci. Fenn., 50 (1922), 1-46.

[20] Pederson, R., A proof for the Bieberbach conjecture for the sixth coefficient, Arch. Rational Mech. Anal., 31 (1968-69), 331-351.

[21] Pederson, R. and Schiffer, M., A proof for the Bieberbach conjecture for the fifth coefficient, Arch. Rational Mech. Anal., 45 (1972), 161-193.

[22] Rani, M., Singh, G., Some Classes Of Schwarzian Functions And Its Coefficient Inequality That Is Sharp, Turk. Jour. Of Computer and Mathematics Education, 11 (2020), 1366-1372.

[23] Rathore, G. S., Singh, G. and Kumawat, L. et.al., Some Subclasses Of A New Class Of Analytic Functions under Fekete-Szegö Inequality, Int. J. of Res. In Adv. Technology, 7(2019)

[24] Rathore, G. S., Singh, G., Fekete – Szegö Inequality for certain subclasses of analytic functions , Journal Of Chemical , Biological And Physical Sciences, 5(2015) ,

[25] Singh, G., Fekete – Szegö Inequality for a new class and its certain subclasses of analytic functions , General Mathematical Notes, 21 (2014),

[26] Singh, G., Fekete – Szegö Inequality for a new class of analytic functions and its subclass, Mathematical Sciences: International Research Journal, 3 (2014),

[27] Singh, G., Construction of Coefficient Inequality For a new Subclass of Class of Starlike Analytic Functions, Russian Journal of Mathematical Research Series, 1 (2015), 9-13.

[28] Singh, G., Introduction of a new class of analytic functions with its Fekete–Szegö Inequality, International Journal of Mathematical Archive, 5 (2014), 30-35.

[29] Singh, G., An Inequality of second and third Coefficients for a Subclass of Starlike Functions Constructed Using nth Derivative, Kaav Int.J. Of Sci. Eng. And Tech., 4 (2017), 206-210.

[30] Singh, G., Fekete–Szegö Inequality for asymptotic subclasses of family of analytic functions, Stochastic Modelling And Applications, 26 (2022),

[31] Singh, G., Coefficient Inequality for Close to Starlike Functions Constructed Using Inverse Starlike Classes, Kaav Int. J. Of Sci. Eng. And Tech., 4 (2017), 177-182.

[32] Singh, G., Coeff. Inequality for a subclass of Starlike functions that is constructed using nth derivative of the functions in the class, Kaav Int. J. Of Sci. Eng. And Tech., 4 (2017), 199-202.

[33] Singh G., Fekete Szegö Inequality For A Complicated Class Of Analytic Functions Approaching To A Class In The Limit Form And Other Class Directly, IJRES, 10 (9), 619-624, 2022

[34] Singh, G., Fekete–Szegö Inequality for functions approaching to a class in the limit form and another class directly, Journal Of Information And Computational Sciences, 12 (4), 2022, 181-186

[35] Singh, G., Garg, J., Coefficient Inequality For A New Subclass Of Analytic Functions, Mathematical Sciences: International Research Journal, 4(2015)

[36] Singh G, Sharma N., Two new subclasses of already defined class of Analytic functions and establishment of their coefficient inequality, NeuroQuantology, 20 (10), 4967-4976, 2022

[37] Singh, G, Singh, Gagan, Fekete–Szegö Inequality For Subclasses Of A New Class Of Analytic Functions , Proceedings Of The World Congress On Engineering , (2014) , .

[38] Singh G, Houque M. A., A specially designed class of univalent functions and establishment of its coefficient inequality, *NeuroQuantology*, 20 (10), 2855-2859, 2022

[39] Singh G, Kaur C, Analytic functions subordinate to leaf-like domain, *Advances in Mechanics*, 10 (1), 1444-1448, 2022

[40] Singh, G, Sarao, M. S., and Mehrok, B. S., Fekete – Szegö Inequality For A New Class Of Analytic Functions , Conference Of Information And Mathematical Sciences , (2013).

[41] Singh. G, Singh. Gagan, Sarao. M. S., Fekete – Szegö Inequality for a New Class of Convex Starlike Analytic Functions, Conf. Of Information and Mathematical Sciences, (2013).

[42] Singh, G., Kaur, G., Coefficient Inequality for a Subclass of Starlike Function generated by symmetric points, *Ganita*, 70 (2020), 17-24.

[43] Singh ,G., Kaur, G., Coefficient Inequality For A New Subclass Of Starlike Functions, *International Journal Of Research In Advent Technology*, 5(2017) ,

[44] Singh, G., Kaur, G., Fekete-Szegö Inequality For A New Subclass Of Starlike Functions, *International Journal Of Research In Advent Technology*, 5(2017) ,

[45] Singh, G., Kaur, G., Fekete-Szegö Inequality For Subclass Of Analytic Function Based On Generalized Derivative, *Aryabhatta Journal Of Mathematics And Informatics*, 9(2017) ,

[46] Singh, G., Kaur, G., Coefficient Inequality For a subclass of analytic function using subordination method with extremal function, *Int. J. Of Advance Res. In Sci&Engg* , 7 (2018)

[47] Singh, G., Kaur, G., Arif, M., Chinram R, Iqbal J, A study of third and fourth Hankel determinant problem for a particular class of bounded turning functions, *Mathematical Problems in Engineering*, 2021

[48] Singh, G. and Kaur, G., 4th Hankel determinant for α bounded turning function, *Advances in Mathematics: Scientific Journal*, 9 (12), 10563-10567

[49] Singh, G., Kaur, N., Fekete-Szegö Inequality For Certain Subclasses Of Analytic Functions, *Mathematical Sciences: International Research Journal*, 4(2015)

[50] Singh G, Patil A. S., An extraordinary class of asymptotic analytical functions with coefficient inequality, *NeuroQuantology*, 20 (10), 4960-4966, 2022

[51] Singh, G, Singh, B, Fekete Szego Coefficient Inequality of Regular Functions for A Special Class, *International Journal of Research in Engineering and Science*, 10(8), 2022, 556-560

[52] Singh, G, Singh, P., Fekete – Szegö Inequality For Functions Belonging To A Certain Class Of Analytic Functions Introduced Using Linear Combination Of Variational Powers Of Starlike And Convex Functions, *Journal Of Positive School Psychology*, 6 (2022), 8387-8391.

[53] Singh. G, Rani M, An advance subclass of Analytic Functions having a unique coefficient inequality, *Int. J. of Research in Engineering and Science*, 10 (8), 2022, 474-476

[54] Singh, G., Singh, G., Singh, G., A subclass of bi-univalent functions defined by generalized Salagean operator related to shell-like curves connected with Fibonacci numbers, *International Journal of Mathematics and Mathematical Sciences*, 2019

[55] Singh, G., Singh, G., Singh, G., A generalized subclass of alpha convex biunivalent functions of complex order, *Jnanabha*, 50 (1), 65-71, 2020

[56] Singh, G., Singh, G., Singh, G., Upper bound on fourth Hankel determinant for certain subclass of multivalent functions, *Jnanabha*, 50 (2), 122-127, 2020

[57] Singh, G., Singh, G., Singh, G., Certain subclasses of univalent and biunivalent functions related to shell-like curves connected with Fibonacci numbers, *General Mathematics*, 28 (1), 125-140, 2020

[58] Singh, G., Singh, G., Singh, G., Certain subclasses of Sakaguchitype bi-univalent functions, *Ganita*, 69 (2), 45-55, 2019

[59] Singh, G., Singh, G., Singh, G., Certain Subclasses of Bi-Close-to-Convex Functions Associated with Quasi-Subordination, *Abstract and Applied Analysis*, 1, 1-6, 2019

[60] Singh, G., Singh, G., Singh, G., Fourth Hankel determinant for a subclass of analytic functions defined by generalized Salagean operator, *Creat. Math. Inform.*, 31(2), 229-240, 2022

[61] Singh, G., Singh, G., Singh, G., Certain subclasses of multivalent functions defined with generalized Salagean operator and related to sigmoid function and lemniscate of Bernoulli, *J. Frac. Calc. Appl.*, 13 (1), 65-81, 2022

[62] Srivastava H. M., G. Kaur, Singh. G, Estimates of fourth Hankel determinant for a class of analytic functions with bounded turnings involving cardioid domains, *Journal of Nonlinear and Convex Analysis*, 22 (3), 511-526, 2021