

# COEFFICIENT INEQUALITY MAKING RESULTS SHARP FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

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## ABSTRACT

We have introduced subclasses of analytic functions and have obtained sharp upper bounds of the Fekete Szego functional  $|a_3 - \mu a_2^2|$  for the analytic function  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ ,  $|z| < 1$  belonging to these classes and subclasses.

**Keywords:** Univalent functions, Starlike functions, Close to convex functions and bounded functions.

## 1. INTRODUCTION

Let  $\mathcal{A}$  denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

analytic in the unit disc given by  $\mathbb{E} = \{z: |z| < 1\}$ . Let  $\mathcal{S}$  be the class of analytic functions of the form (1.1), which are univalent in  $\mathbb{E}$ . In 1916, Bieber Bach ([1], [2]) proved that  $|a_2| \leq 2$  for the functions  $f(z) \in \mathcal{S}$ . In 1923, Löwner [10] proved that  $|a_3| \leq 3$  for the functions  $f(z) \in \mathcal{S}$ .

With the known estimates  $|a_2| \leq 2$  and  $|a_3| \leq 3$ , it was natural to seek some relation between  $a_3$  and  $a_2^2$  for the class  $\mathcal{S}$ . Fekete and Szegő [4] used Löwner's method to prove the following well known result for the class  $\mathcal{S}$ .

Let  $f(z) \in \mathcal{S}$ , then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq 0; \\ 1 + 2 \exp\left(\frac{-2\mu}{1-\mu}\right), & \text{if } 0 \leq \mu \leq 1; \\ 4\mu - 3, & \text{if } \mu \geq 1. \end{cases} \quad (1.2)$$

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes  $\mathcal{S}([3], [9])$ .

Let us define some subclasses of  $\mathcal{S}$ .

We denote by  $S^*$ , the class of univalent starlike functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A} \text{ and satisfying the condition}$$

$$\operatorname{Re} \left( \frac{zg(z)}{g(z)} \right) > 0, z \in \mathbb{E}. \quad (1.3)$$

We denote by  $\mathcal{K}$ , the class of univalent convex functions and satisfying the condition

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in \mathcal{A}$$

$$\operatorname{Re} \left( \frac{zh'(z)}{h'(z)} \right) > 0, z \in \mathbb{E}. \quad (1.4)$$

A function  $f(z) \in \mathcal{A}$  is said to be close to convex if there exists  $g(z) \in S^*$  such that

$$\operatorname{Re} \left( \frac{zf'(z)}{g(z)} \right) > 0, z \in \mathbb{E}. \quad (1.5)$$

The class of close to convex functions is denoted by  $\mathcal{C}$  and was introduced by Kaplan [7] and it was shown by him that all close to convex functions are univalent.

$$S^*(A, B) = \left\{ f(z) \in \mathcal{A}; \frac{zf'(z)}{f(z)} < \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E} \right\} \quad (1.6)$$

$$\mathcal{K}(A, B) = \left\{ f(z) \in \mathcal{A}; \left( \frac{zf'(z)}{f'(z)} \right)' < \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E} \right\} \quad (1.7)$$

It is obvious that  $S^*(A, B)$  is a subclass of  $S^*$  and  $\mathcal{K}(A, B)$  is a subclass of  $\mathcal{K}$ .

Several authors studied and introduced various classes and subclasses of univalent analytic functions and established Fekete Szego inequality for the same. ([3]-[9], [12]-[15], [22]-[62])

N. Kaur [11] introduced a new subclass as and have established its coefficient inequality.

$$S^*(f, f', \alpha, \beta) = \left\{ f(z) \in \mathcal{A}; (1 - \alpha) \left( \frac{zf'(z)}{f(z)} \right)^\beta + \alpha \left( \frac{(zf'(z))'}{f'(z)} \right)^{1-\beta} < \frac{1+z}{1-z}; z \in \mathbb{E} \right\}$$

We will deal with the subclass of  $S^*(f, f', \alpha, \beta)$  defined as follows in the present paper:

$$S^*(f, f', \alpha, \beta, A, B) = \left\{ f(z) \in \mathcal{A}; (1 - \alpha) \left( \frac{zf'(z)}{f(z)} \right)^\beta + \alpha \left( \frac{(zf'(z))'}{f'(z)} \right)^{1-\beta} < \frac{1+Az}{1+Bz}; z \in \mathbb{E} \right\} \quad (1.8)$$

We will deal with the subclass  $S^*(f, f', \alpha, \beta, \delta)$  defined as follows in our next paper:

$$S^*(f, f', \alpha, \beta, \delta) = \left\{ f(z) \in \mathcal{A}; (1 - \alpha) \left( \frac{zf'(z)}{f(z)} \right)^\beta + \alpha \left( \frac{(zf'(z))'}{f'(z)} \right)^{1-\beta} < \left( \frac{1+z}{1-z} \right)^\delta; z \in \mathbb{E} \right\} \quad (1.9)$$

Symbol  $<$  stands for subordination, which we define as follows:

**Principle of Subordination:** Let  $f(z)$  and  $F(z)$  be two functions analytic in  $\mathbb{E}$ . Then  $f(z)$  is called subordinate to  $F(z)$  in  $\mathbb{E}$  if there exists a function  $w(z)$  analytic in  $\mathbb{E}$  satisfying the conditions  $w(0) = 0$  and  $|w(z)| < 1$  such that  $f(z) = F(w(z))$ ;  $z \in \mathbb{E}$  and we write  $f(z) < F(z)$ .

By  $\mathcal{U}$ , we denote the class of analytic bounded functions of the form  $w(z) = \sum_{n=1}^{\infty} d_n z^n$ ,  $w(0) = 0$ ,  $|w(z)| < 1$ .

$$(1.10)$$

It is known that  $|d_1| \leq 1$ ,  $|d_2| \leq 1 - |d_1|^2$ . (1.11)

## PRELIMINARY LEMMAS:

For  $0 < c < 1$ , we write  $w(z) = \left( \frac{c+z}{1+cZ} \right)$  so that

$$\frac{1+w(z)}{1-w(z)} = 1 + 2cz + 2z^2 + \dots \quad (2.1)$$

## 2. MAIN RESULTS

### THEORE 2.1

Let  $f(z) \in S^*(f, f', \alpha, \beta, A, B)$ , then The results are sharp.

$$\begin{aligned} & \left| a_3 - \mu a_2^2 \right| \\ & \leq \begin{cases} \text{if } \mu \leq \frac{(A-B)^2(8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta)}{4(3\alpha+\beta-4\alpha\beta)\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} - \frac{(A-B)^2}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} \mu, \\ \frac{(A-B)8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta-4\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{(3\alpha+\beta-4\alpha\beta)}; \\ \text{if } \frac{(A-B)}{2(3\alpha+\beta-4\alpha\beta)} \frac{(A-B)8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta-4\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{(3\alpha+\beta-4\alpha\beta)} \leq \mu \leq \\ \frac{4\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2 - (A-B)8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)}; \\ \text{if } \mu \geq \frac{(A-B)^2}{4\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} \mu - \frac{(A-B)^2(8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta)}{4(3\alpha+\beta-4\alpha\beta)\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}, \\ \frac{4\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2 - (A-B)8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)} \end{cases} \end{aligned} \quad (3.1)$$

**Proof 2.2.** By definition of  $S^*(f, f', \alpha, \beta, A, B)$ , we have

$$(1 - \alpha) \left( \frac{zf'(z)}{f(z)} \right)^\beta + \alpha \left( \frac{(zf'(z))'}{f'(z)} \right)^{1-\beta} = \frac{1+Aw(z)}{1+Bw(z)}; w(z) \in \mathcal{U}. \quad (3.4)$$

Expanding the series (3.4), we get

$$(1 - \alpha) \left\{ 1 + \beta a_2 z + (2\beta a_3 + \frac{\beta(\beta-3)}{2} a_2^2) z^2 + \dots \right\} + \alpha \left\{ 1 + 2(1 - \beta) a_2 z + 2(1 - \beta)(3a_3 - (\beta + 2)a_2^2) z^2 + \dots \right\} = (1 + (A - B)c_1 z + (A - B)(c_2 - Bc_1^2) z^2 + \dots). \quad (3.5)$$

Identifying terms in (3.5), we get

$$a_2 = \frac{(A-B)}{(1-\alpha)\beta+2\alpha(1-\beta)} c_1 \quad (3.6)$$

$$a_3 = \frac{(A-B)}{2(3\alpha+\beta-4\alpha\beta)} c_2 + \frac{(A-B)^2(8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta)}{4(3\alpha+\beta-4\alpha\beta)\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} c_1^2. \quad (3.7)$$

From (3.6) and (3.7), we obtain

$$a_3 - \mu a_2^2 = \frac{(A-B)}{2(3\alpha+\beta-4\alpha\beta)} c_2 + \left[ \frac{(A-B)^2(8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta)}{4(3\alpha+\beta-4\alpha\beta)\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} - \frac{(A-B)^2}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} \mu \right] c_1^2. \quad (3.8)$$

Taking absolute value, (3.8) can be rewritten as

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{3\alpha+\beta-4\alpha\beta} |c_2| + \frac{(A-B)^2}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} \left| \frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)} - \mu \right| |c_1|^2. \quad (3.9)$$

Using (1.9) in (3.9), we get

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{(A-B)}{3\alpha+\beta-4\alpha\beta} (1 - |c_1|^2) \\ &\quad + \frac{(A-B)^2}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} \left| \frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)} - \mu \right| |c_1|^2 \\ &= \frac{(A-B)}{3\alpha+\beta-4\alpha\beta} + \frac{(A-B)^2}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} \left[ \left| \frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)} - \mu \right| - \frac{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{(A-B)(3\alpha+\beta-4\alpha\beta)} \right] |c_1|^2. \end{aligned} \quad (3.10)$$

$$\text{Case I: } \mu \leq \frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)}.$$

(3.10) can be rewritten as

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{3\alpha+\beta-4\alpha\beta} + \frac{(A-B)^2}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} \left[ \frac{(A-B)8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta-4\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{(3\alpha+\beta-4\alpha\beta)} - \mu \right] |c_1|^2. \quad (3.11)$$

$$\text{Subcase I (a): } \mu \leq \frac{(A-B)8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta-4\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{(3\alpha+\beta-4\alpha\beta)}.$$

Using (1.9), (3.11) becomes

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)^2(8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta)}{4(3\alpha+\beta-4\alpha\beta)\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} - \frac{(A-B)^2}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} \mu \quad (3.12)$$

$$\text{Subcase I (b): } \mu \geq \frac{(A-B)8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta-4\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{(3\alpha+\beta-4\alpha\beta)}.$$

We obtain from (3.11)

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{3\alpha+\beta-4\alpha\beta}. \quad (3.13)$$

$$\text{Case II: } \mu \geq \frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)}$$

Proceeding as in case I, we get

$$|a_3 - \mu a_2^2| \leq \frac{1}{3\alpha+\beta-4\alpha\beta} + \frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} \left[ \mu - \frac{4\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2 - (A-B)8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)} \right] |c_1|^2. \quad (3.14)$$

$$\text{Subcase II (a): } \mu \leq \frac{4\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2 - (A-B)8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)}$$

(3.14) takes the form

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{3\alpha+\beta-4\alpha\beta} \quad (3.15)$$

Combining subcase I (b) and subcase II (a), we obtain

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{(A-B)}{3\alpha+\beta-4\alpha\beta} \text{ if } \frac{(A-B)8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta-4\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{(3\alpha+\beta-4\alpha\beta)} \leq \mu \leq \\ &\frac{4\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2 - (A-B)8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)} \end{aligned} \quad (3.16)$$

$$\text{Subcase II (b): } \mu \geq \frac{4\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2 - (A-B)8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)}$$

Proceeding as in subcase I (a), we get

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)^2}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} \mu - \frac{(A-B)^2(8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta)}{4(3\alpha+\beta-4\alpha\beta)\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} \quad (3.17)$$

Combining (3.12), (3.16) and (3.17), the theorem is proved.

**Corollary 2.3:** Putting  $\alpha = 1, \beta = 0, A = 1, B = -1$  in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} 1 - \mu, & \text{if } \mu \leq 1; \\ \frac{1}{3} & \text{if } 1 \leq \mu \leq \frac{4}{3}; \\ \mu - 1, & \text{if } \mu \geq \frac{4}{3} \end{cases}$$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent convex functions.

**Corollary 2.3:** Putting  $\alpha = 0, \beta = 1, A = 1, B = -1$  in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq \frac{1}{2}; \\ 1 & \text{if } \frac{1}{2} \leq \mu \leq 1; \\ 4\mu - 3, & \text{if } \mu \geq 1 \end{cases}$$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent starlike functions.

**Corollary 2.4:** Putting  $A = 1, B = -1$  in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{\{(1-\alpha)\beta + 2\alpha(1-\beta)\}^2} \left[ \frac{8\alpha + 3\beta + 4\alpha^2 - 12\alpha^2\beta - 9\alpha\beta^2 - 7\alpha\beta}{(3\alpha + \beta - 4\alpha\beta)} - 4\mu \right], \\ \text{if } \mu \leq \frac{8\alpha + 3\beta + 4\alpha^2 - \beta^2 - 3\alpha\beta^2 - 7\alpha\beta}{4(3\alpha + \beta - 4\alpha\beta)}; \\ \frac{1}{3\alpha + \beta - 4\alpha\beta} \\ \text{if } \frac{8\alpha + 3\beta + 4\alpha^2 - \beta^2 - 3\alpha\beta^2 - 7\alpha\beta}{4(3\alpha + \beta - 4\alpha\beta)} \leq \mu \leq \frac{8\alpha + 3\beta + 8\alpha^2 + \beta^2 - 24\alpha^2\beta - 6\alpha\beta^2 - 7\alpha\beta}{4(3\alpha + \beta - 4\alpha\beta)}; \\ \frac{1}{\{(1-\alpha)\beta + 2\alpha(1-\beta)\}^2} \left[ 4\mu - \frac{8\alpha + 3\beta + 4\alpha^2 - 12\alpha^2\beta - 9\alpha\beta^2 - 7\alpha\beta}{(3\alpha + \beta - 4\alpha\beta)} \right], \\ \text{if } \mu \geq \frac{8\alpha + 3\beta + 8\alpha^2 + \beta^2 - 24\alpha^2\beta - 6\alpha\beta^2 - 7\alpha\beta}{4(3\alpha + \beta - 4\alpha\beta)} \end{cases}$$

These estimates were derived by N. Kaur [11] and are results for the subclass  $S^*(f, f', \alpha, \beta)$  of univalent starlike functions.

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