

ON APPROXIMATION BY SOME INTEGRAL MODIFICATION OF JAKIMOVSKI-LEVIATAN OPERATORS

Nandita Gupta¹

¹Government Polytechnic, Mahasamund, Chhattisgarh, India.

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ABSTRACT

In the present paper we propose an integral modification of Jakimovski- Leviatan operators involving Appell Polynomials. Using Korovkin Theorem we obtain the approximation properties of these operators. We compute an estimate of the order of approximation of a continuous functions by means of the operator via first order modulus of continuity. We also give an asymptotic estimate through Voronovskaja - type result for these operators.

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Key Words- Linear positive operators, Appell Polynomials, Order of approximation, Modulus of continuity, Voronovskaja - type theorem .

1. INTRODUCTION

Jakimovski and Leviatan [3] introduced a new type of operators P_n by using Appell polynomials as follows,

$$P_n(f; x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right) \quad (1.1)$$

for $f \in E[0, \infty)$. B. Wood proved in [6] that the operators P_n are positive on $[0, \infty)$

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if and only if $\sum_{n=0}^{\infty} a_n \geq 0$, $n \in \mathbb{N}$

$g(1) \neq 0$

Recall that Appell polynomials are polynomials defined as follows: Let

$$g(u) = \sum_{n=0}^{\infty} a_n u^n, g(1) \neq 0$$

be an analytic function in the disc $|u| < r$, ($r > 1$) and

$$p_k(x) = \sum_{i=0}^k a_i \frac{x^{k-i}}{(k-i)!}, (k \in \mathbb{N})$$

be the Appell polynomials defined by the identity

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x) u^k \quad (1.2)$$

Ibrahim Bu"yu"kyazici et. al. [2] gave a Chlodowsky type generalization of Jakimovski - Leviatan operators given by

$$P_n^*(f; x) = \frac{e^{-\frac{n}{b_n}x}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) f\left(\frac{k}{n}b_n\right) \quad (1.3)$$

with b_n a positive increasing sequence with the properties

$$\lim_{n \rightarrow \infty} b_n = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{b_n}{n} = 0 \quad (1.4)$$

Motivated by the operators given in (1.2) and (1.3) we consider the following integral modification of the operator (1.3):

$$A_n(f; x) = \frac{n}{b_n} \frac{e^{-\frac{n}{b_n}x}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \int_0^{\infty} \frac{e^{-\frac{n}{b_n}t}}{k!} \left(\frac{n}{b_n}t\right)^k f(t) dt \quad (1.5)$$

where b_n is a positive increasing sequence satisfying (1.4). We see by construction of the operator A_n that the condition for positivity given by Wood for operator P_n is applicable here also. So throughout this paper we will assume that the operators A_n are positive.

Approximation properties of $A_n(f; x)$

We denote by $CE[0, \infty)$ the set of all continuous functions f on $[0, \infty)$ with the property that $|f(x)| \leq \beta e^{\alpha x}$ for all $x \geq 0$ and some positive finite α and β .

Following lemma was given in [3].

Lemma 2.1. From (1.2) we have

$$\begin{aligned} \sum_{k=0}^{\infty} p_k \left(\frac{n}{b_n} x \right) &= g(1) e^{\frac{n}{b_n} x} = \phi_0(x) \\ \sum_{k=0}^{\infty} k p_k \left(\frac{n}{b_n} x \right) &= \left(\frac{n}{b_n} x g(1) + g'(1) \right) e^{\frac{n}{b_n} x} = \phi_1(x) \\ \sum_{k=0}^{\infty} k^2 p_k \left(\frac{n}{b_n} x \right) &= \left(g(1) \frac{n^2}{b_n^2} x^2 + (g(1) + 2g'(1)) \frac{n}{b_n} x + g'(1) + g''(1) \right) e^{\frac{n}{b_n} x} = \phi_2(x) \\ \sum_{k=0}^{\infty} k^3 p_k \left(\frac{n}{b_n} x \right) &= \left(g(1) \frac{n^3}{b_n^3} x^3 + (4g(1) + 3g'(1)) \frac{n^2}{b_n^2} x^2 + (g(1) + 8g'(1) + 3g''(1)) \frac{n}{b_n} x \right. \\ &\quad \left. + g'(1) + 4g''(1) + g'''(1) \right) e^{\frac{n}{b_n} x} = \phi_3(x) \\ \sum_{k=0}^{\infty} k^4 p_k \left(\frac{n}{b_n} x \right) &= \left(g(1) \frac{n^4}{b_n^4} x^4 + (10g(1) + 4g'(1)) \frac{n^3}{b_n^3} x^3 + (14g(1) + 30g'(1) + 6g''(1)) \frac{n^2}{b_n^2} x^2 \right. \\ &\quad \left. + (g(1) + 28g'(1) + 30g''(1) + 4g'''(1)) \frac{n}{b_n} x + g'(1) + 14g''(1) + 10g'''(1) \right. \\ &\quad \left. + g^{(4)}(1) \right) e^{\frac{n}{b_n} x} = \phi_4(x) \end{aligned}$$

Using lemma (2.1) and eq. (1.5) we get following results.

Lemma 2.2. The operators A_n defined by eq. (1.5) satisfy,

$$\begin{aligned} A_n(e_0; x) &= 1 \\ A_n(e_1; x) &= x + \frac{b_n}{n} \left(1 + \frac{g'(1)}{g(1)} \right) \\ A_n(e_2; x) &= x^2 + \frac{b_n}{n} \frac{(4g(1) + 2g'(1))}{g(1)} x + \left(\frac{b_n}{n} \right)^2 \frac{(2g(1) + 4g'(1) + g''(1))}{g(1)} \\ A_n(e_3; x) &= x^3 + \left(\frac{b_n}{n} \right) \frac{(10g(1) + 3g'(1))}{g(1)} x^2 + \left(\frac{b_n}{n} \right)^2 \frac{(18g(1) + 20g'(1) + 3g''(1))}{g(1)} x \\ &\quad + \left(\frac{b_n}{n} \right)^3 \frac{(6g(1) + 18g'(1) + 10g''(1) + g'''(1))}{g(1)} \\ A_n(e_4; x) &= x^4 + \left(\frac{b_n}{n} \right) \frac{(20g(1) + 4g'(1))}{g(1)} x^3 + \left(\frac{b_n}{n} \right)^2 \frac{(89g(1) + 60g'(1) + 6g''(1))}{g(1)} x^2 \\ &\quad + \left(\frac{b_n}{n} \right)^3 \frac{(96g(1) + 178g'(1) + 60g''(1) + 4g'''(1))}{g(1)} x \\ &\quad + \left(\frac{b_n}{n} \right)^4 \frac{(24g(1) + 96g'(1) + 89g''(1) + 20g'''(1) + g^{(4)}(1))}{g(1)} \end{aligned}$$

where $e_i(t) = t^i$, $i = 0, 1, 2, 3, 4$.

Proof. From the definition of $A_n(f; x)$ and lemma (2.1), we have

$$\begin{aligned} A_n(e_0; x) &= \frac{n}{b_n} \frac{e^{-\frac{n}{b_n}x}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \int_0^{\infty} \frac{e^{-\frac{n}{b_n}t}}{k!} \left(\frac{n}{b_n}t\right)^k dt \\ &= \frac{e^{-\frac{n}{b_n}x}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \frac{\Gamma(k+1)}{k!} = \frac{e^{-\frac{n}{b_n}x}}{g(1)} \phi_0(x) = 1 \end{aligned}$$

$$\begin{aligned} A_n(e_1; x) &= \frac{n}{b_n} \frac{e^{-\frac{n}{b_n}x}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \int_0^{\infty} \frac{e^{-\frac{n}{b_n}t}}{k!} \left(\frac{n}{b_n}t\right)^k t dt \\ &= \left(\frac{b_n}{n}\right) \frac{e^{-\frac{n}{b_n}x}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \frac{\Gamma(k+2)}{k!} = \left(\frac{b_n}{n}\right) \frac{e^{-\frac{n}{b_n}x}}{g(1)} (\phi_1(x) + \phi_0(x)) \\ &= x + \frac{b_n}{n} \left(1 + \frac{g'(1)}{g(1)}\right) \end{aligned}$$

$$\begin{aligned} A_n(e_2; x) &= \frac{n}{b_n} \frac{e^{-\frac{n}{b_n}x}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \int_0^{\infty} \frac{e^{-\frac{n}{b_n}t}}{k!} \left(\frac{n}{b_n}t\right)^k t^2 dt \\ &= \left(\frac{b_n}{n}\right)^2 \frac{e^{-\frac{n}{b_n}x}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \frac{\Gamma(k+3)}{k!} \\ &= \left(\frac{b_n}{n}\right)^2 \frac{e^{-\frac{n}{b_n}x}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) (k^2 + 3k + 2) \\ &= \left(\frac{b_n}{n}\right)^2 \frac{e^{-\frac{n}{b_n}x}}{g(1)} (\phi_2(x) + 3\phi_1(x) + 2\phi_0(x)) \\ &= x^2 + \frac{b_n}{n} \frac{(4g(1) + 2g'(1))}{g(1)} x + \left(\frac{b_n}{n}\right)^2 \frac{(2g(1) + 4g'(1) + g''(1))}{g(1)} \end{aligned}$$

For e_3 we have

$$\begin{aligned} A_n(e_3; x) &= \frac{n}{b_n} \frac{e^{-\frac{n}{b_n}x}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \int_0^{\infty} \frac{e^{-\frac{n}{b_n}t}}{k!} \left(\frac{n}{b_n}t\right)^k t^3 dt \\ &= \left(\frac{b_n}{n}\right)^3 \frac{e^{-\frac{n}{b_n}x}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \frac{\Gamma(k+4)}{k!} \\ &= \left(\frac{b_n}{n}\right)^3 \frac{e^{-\frac{n}{b_n}x}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) (k^3 + 6k^2 + 11k + 6) \\ &= \left(\frac{b_n}{n}\right)^3 \frac{e^{-\frac{n}{b_n}x}}{g(1)} (\phi_3(x) + 6\phi_2(x) + 11\phi_1(x) + 6\phi_0(x)) \\ &= x^3 + \left(\frac{b_n}{n}\right) \frac{(10g(1) + 3g'(1))}{g(1)} x^2 + \left(\frac{b_n}{n}\right)^2 \frac{(18g(1) + 20g'(1) + 3g''(1))}{g(1)} x \\ &\quad + \left(\frac{b_n}{n}\right)^3 \frac{(6g(1) + 18g'(1) + 10g''(1) + g'''(1))}{g(1)} \end{aligned}$$

and

$$\begin{aligned}
 A_n(e_4; x) &= \frac{n}{b_n} \frac{e^{-\frac{n}{b_n}x}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \int_0^{\infty} \frac{e^{-\frac{n}{b_n}t}}{k!} \left(\frac{n}{b_n}t\right)^k t^4 dt \\
 &= \left(\frac{b_n}{n}\right)^4 \frac{e^{-\frac{n}{b_n}x}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \frac{\Gamma(k+5)}{k!} \\
 &= \left(\frac{b_n}{n}\right)^4 \frac{e^{-\frac{n}{b_n}x}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) (k^4 + 10k^3 + 35k^2 + 50k + 24) \\
 &= \left(\frac{b_n}{n}\right)^4 \frac{e^{-\frac{n}{b_n}x}}{g(1)} (\phi_4(x) + 10\phi_3(x) + 35\phi_2(x) + 50\phi_1(x) + 24\phi_0(x)) \\
 &= x^4 + \left(\frac{b_n}{n}\right) \frac{(20g(1) + 4g'(1))}{g(1)} x^3 + \left(\frac{b_n}{n}\right)^2 \frac{(89g(1) + 60g'(1) + 6g''(1))}{g(1)} x^2 \\
 &\quad + \left(\frac{b_n}{n}\right)^3 \frac{(96g(1) + 178g'(1) + 60g''(1) + 4g'''(1))}{g(1)} x \\
 &\quad + \left(\frac{b_n}{n}\right)^4 \frac{(24g(1) + 96g'(1) + 89g''(1) + 20g'''(1) + g^{(4)}(1))}{g(1)}
 \end{aligned}$$

The following result can be easily derived from the previous lemma.

Lemma 2.3. The operators A_n defined by eq. (1.5) satisfy,

$$A_n(t - x; x) = \frac{b_n}{n} \left(1 + \frac{g'(1)}{g(1)}\right) \quad (2.1)$$

$$A_n((t - x)^2; x) = 2x \frac{b_n}{n} + \left(\frac{b_n}{n}\right)^2 \frac{(2g(1) + 4g'(1) + g''(1))}{g(1)} \quad (2.2)$$

$$\begin{aligned}
 A_n((t - x)^4; x) &= \left(\frac{b_n}{n}\right)^2 \frac{(29g(1) + 4g'(1))}{g(1)} x^2 + \left(\frac{b_n}{n}\right)^3 \frac{(72g(1) + 106g'(1) + 20g''(1))}{g(1)} x \\
 &\quad + \left(\frac{b_n}{n}\right)^4 \frac{(24g(1) + 96g'(1) + 89g''(1) + 20g'''(1) + g^{(4)}(1))}{g(1)}
 \end{aligned} \quad (2.3)$$

Theorem 2.4. For $f \in CE[0, \infty)$, the operators A_n converge uniformly to f on $[0, a]$, $a > 0$ as $n \in \mathbb{N}$.

Proof. From lemma (2.2) we have

$$\lim_{n \rightarrow \infty} A_n(e_i; x) = e_i(x), \quad i = 0, 1, 2.$$

$n \rightarrow \infty$

On applying Korovkin theorem [1] we get the desired result.

Order Of Approximation

In this section we give an estimate of the order of approximation of a function $f \in CE[0, \infty)$ by means of the operator A_n , using the first order modulus of continuity.

Let $f \in C[0, b]$. The modulus of continuity of f denoted by $\omega(f, \delta)$, is defined to be

$$\omega(f, \delta) = \sup_{|s-x| < \delta, s, x \in [0, b]} |f(s) - f(x)|$$

The modulus of continuity of the function f in $C[0, b]$ gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$.

It is well known that for any $\delta > 0$ and each $s \in [0, b]$

$$|f(s) - f(x)| \leq \omega(f, \delta) \left(1 + \frac{|s - x|}{\delta}\right) \quad (3.1)$$

Theorem 3.1. For $f \in CE[0, \infty)$ and $x \in [0, a]$, $a > 0$ we have

$$|A_n(f; x) - f(x)| \leq \left[1 + \sqrt{2a + \frac{b_n(2g(1) + 4g'(1) + g''(1))}{n g(1)}}\right]$$

Proof. By using eq. (3.1) linearity of operators A_n we obtain

$$\begin{aligned} |A_n(f; x) - f(x)| &\leq A_n(|f(t) - f(x)|; x) \\ &\leq \omega(f, \delta) A_n\left(1 + \frac{|t - x|}{\delta}; x\right) \\ &= \omega(f, \delta) \left(1 + \frac{1}{\delta} \left(\frac{n}{b_n} \frac{e^{-\frac{n}{b_n}x}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \int_0^{\infty} \frac{e^{-\frac{n}{b_n}t}}{k!} \left(\frac{n}{b_n}t\right)^k |t - x| dt\right)\right) \end{aligned}$$

Using the Cauchy - Schwarz inequality we obtain

$$\begin{aligned} |A_n(f; x) - f(x)| &\leq \omega(f, \delta) \left(1 + \frac{1}{\delta} \left(\frac{n}{b_n} \frac{e^{-\frac{n}{b_n}x}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \int_0^{\infty} \frac{e^{-\frac{n}{b_n}t}}{k!} \left(\frac{n}{b_n}t\right)^k (t - x)^2 dt\right)^{1/2}\right) \\ &= \omega(f, \delta) \left(1 + \frac{1}{\delta} \sqrt{A_n((t - x)^2; x)}\right) \end{aligned} \quad (3.2)$$

From eq. (2.2), for $x \in [0, a]$, $a > 0$ we obtain

$$A_n((t - x)^2; x) \leq 2a \frac{b_n}{n} + \left(\frac{b_n}{n}\right)^2 \frac{(2g(1) + 4g'(1) + g''(1))}{g(1)} \quad (3.3)$$

Taking $\delta = \sqrt{\frac{b_n}{n}}$ and using eq. (3.3) in eq. (3.2) we get the desired result.

4. Theorem of Voronovskaja - type

Now we give a Voronovskaja - type relation for the operator A_n .

Theorem 4.1. If $f \in C^2[0, \infty)$ then

$$\lim_{n \rightarrow \infty} \frac{n}{b_n} \{A_n(f; x) - f(x)\} = f'(x) \left[1 + \frac{g'(1)}{g(1)}\right] + x f''(x).$$

uniformly for $x \in [0, a]$, $a > 0$.

Proof. For a fixed $x_0 \in [0, \infty)$, by Taylor's formula we have for every $t \in [0, \infty)$

$$f(t) = f(x_0) + f'(x_0)(t - x_0) + \frac{1}{2} f''(x_0)(t - x_0)^2 + \psi(t; x_0)(t - x_0)^2, \quad (4.1)$$

where $\psi(t; x_0)$ is a function belonging to the space $CE[0, \infty)$ and $\lim_{t \rightarrow x_0} \psi(t; x_0) = 0$. Then by (4.1) and lemma (2.2) we can write for every $n \in \mathbb{N}$,

$$\begin{aligned} \frac{n}{b_n} [A_n(f; x_0) - f(x_0)] &= \frac{n}{b_n} f'(x_0) A_n(t - x_0; x_0) + \frac{1}{2} \frac{n}{b_n} f''(x_0) A_n((t - x_0)^2; x_0) \\ &\quad + \frac{n}{b_n} A_n(\psi(t; x_0)(t - x_0)^2; x_0) \end{aligned} \quad (4.2)$$

From eqs. (2.1) and (2.3) we have

$$\lim_{n \rightarrow \infty} \frac{n}{b_n} A_n(t - x_0; x_0) = \left(1 + \frac{g'(1)}{g(1)}\right) \quad (4.3)$$

$$\lim_{n \rightarrow \infty} \frac{n}{b_n} A_n((t - x_0)^2; x_0) = 2x_0 \quad (4.4)$$

By Cauchy-Schwarz inequality we get for $n \in \mathbb{N}$

$$\left| \frac{n}{b_n} A_n(\psi(t; x_0)(t - x_0)^2; x_0) \right| \leq \{A_n(\psi^2(t; x_0); x_0)\}^{\frac{1}{2}} \left\{ \left(\frac{n}{b_n}\right)^2 A_n((t - x_0)^4; x_0) \right\}^{\frac{1}{2}} \quad (4.5)$$

From (2.3) we have

$$\lim_{n \rightarrow \infty} \left(\frac{n}{b_n}\right)^2 A_n((t - x_0)^4; x_0) = \frac{(29g(1) + 4g'(1))}{g(1)} x_0^2 \quad (4.6)$$

Let $\phi(t, x_0) = \psi^2(t; x_0)$, $t \geq 0$. Then $\phi(t, x_0) \in CE[0, \infty)$ and $\lim_{t \rightarrow x_0} \phi(t, x_0) = 0$

Then from Theorem (2.4) we have

$$\lim_{n \rightarrow \infty} A_n(\psi^2(t; x_0); x_0) = \lim_{n \rightarrow \infty} A_n(\phi(t; x_0); x_0) = \phi(x_0; x_0) = 0 \quad (4.7)$$

uniformly with respect to $x_0 \in [0, a]$. So by eqs. (4.6)-(4.7) we have

$$\lim_{n \rightarrow \infty} \frac{n}{b_n} A_n(\psi(t; x_0)(t - x_0)^2; x_0) = 0 \quad (4.8)$$

then, taking the limit as $\lim_{n \rightarrow \infty}$ in (4.2) and using (4.2), (4.4) and (4.5) we have

$$\lim_{n \rightarrow \infty} \frac{n}{b_n} \{A_n(f; x) - f(x)\} = f'(x) \left[1 + \frac{g'(1)}{g(1)}\right] + x f''(x).$$

Hence the theorem.

2. CONCLUSION

In the present paper we considered a Chlodovski type integral modification of Jakimovski- Leviatan operators involving Appell Polynomials. We proved that these new operators are approximation process. We have also given an asymptotic estimate through Voronovskaja - type result for these operators.

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