

ON FINDING INTEGRAL SOLUTIONS OF TERNARY QUADRATIC EQUATION

$$x^2 + y^2 = z^2 - 10$$

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ABSTRACT

This paper illustrates the process of obtaining different sets of non-zero distinct integer solutions to the non-homogeneous ternary quadratic Diophantine equations given by $x^2 + y^2 = z^2 - 10$.

Keywords: Non-Homogeneous Quadratic , Ternary Quadratic, Integer Solutions.

1. INTRODUCTION

It is known that Diophantine equations with multidegree and multiple variables are rich in variety[1,2]. While searching for the collection of second degree equations with three unknowns, the authors came across the papers [3-12] in which the authors obtained integer solutions to the ternary quadratic equations $x^2 + y^2 = z^2 + N$, $N = 1, \pm 4, \pm 8, 12, -2k^2, 10, -12$. The above papers motivated us for obtaining non zero distinct integer solutions to the above equation for other values to N . This communication illustrates process of obtaining different sets of non-zero distinct integer solutions to the non-homogeneous ternary quadratic Diophantine equation given by $x^2 + y^2 = z^2 - 10$.

2. METHOD OF ANALYSIS

The non-homogeneous ternary quadratic Diophantine equation under consideration is

$$x^2 + y^2 = z^2 - 10 \quad (1)$$

The process of obtaining different sets of integer solutions to (1) is illustrated below:

Illustration 1:

The choice

$$z = x + h, h \geq 0 \quad (2)$$

in (1) leads to the parabola

$$y^2 = 2hx + h^2 - 10 \quad (3)$$

It is possible to choose h, x so that the R.H.S. of (3) is a perfect square and the value of y is obtained. Substituting the values of h, x in (2), the corresponding value of z satisfying (1) is obtained. For simplicity and brevity, a few examples are given in Table 1 below:

Table 1 : Examples

h	x	y	z
1	$2k^2 - 2k + 5$	$2k - 1$	$2k^2 - 2k + 6$
5	$10k^2 - 10k + 1$	$10k - 5$	$10k^2 - 10k + 6$

Illustration 2:

The substitution of the linear transformations

$$z = (2k^2 - 2k + 6)s, x = (2k^2 - 2k + 5)s \quad (4)$$

in (1) leads to the negative pell equation

$$y^2 = (4k^2 - 4k + 11)s^2 - 10 \quad (5)$$

for which the integer solutions exist when k takes particular values.

Example :1

Considering the value of k to be 1 in (4), it gives the negative pell equation

$$y^2 = 11s^2 - 10 \quad (6)$$

After some algebra, the corresponding integer solutions to (6) are given by

$$y_{n+1} = \left(\frac{1}{2}f_n + \frac{\sqrt{11}}{2}g_n\right) \quad (7)$$

$$s_{n+1} = \left(\frac{1}{2}f_n + \frac{1}{2\sqrt{11}}g_n\right) \quad (8)$$

where $f_n = (10 + 3\sqrt{11})^{n+1} + (10 - 3\sqrt{11})^{n+1}$, $g_n = (10 + 3\sqrt{11})^{n+1} - (10 - 3\sqrt{11})^{n+1}$

Using (8) in (4), one obtains that

$$x_{n+1} = 5\left(\frac{1}{2}f_n + \frac{1}{2\sqrt{11}}g_n\right), z_{n+1} = 6\left(\frac{1}{2}f_n + \frac{1}{2\sqrt{11}}g_n\right) \quad (9)$$

Thus, (7) and (9) represent the integer solutions to (1).

Example :2

Considering the value of k to be 2 in (4), it gives the negative pell equation

$$y^2 = 19s^2 - 10 \quad (10)$$

After some algebra, the corresponding integer solutions to (10) are given by

$$y_{n+1} = \left(\frac{3}{2}f_n + \frac{\sqrt{19}}{2}g_n\right) \quad (11)$$

$$s_{n+1} = \left(\frac{1}{2}f_n + \frac{3g_n}{2\sqrt{19}}\right) \quad (12)$$

where $f_n = (170 + 39\sqrt{19})^{n+1} + (170 - 39\sqrt{19})^{n+1}$, $g_n = (170 + 39\sqrt{19})^{n+1} - (170 - 39\sqrt{19})^{n+1}$

Using (12) in (4), one obtains that

$$x_{n+1} = 9\left(\frac{1}{2}f_n + \frac{3g_n}{2\sqrt{19}}\right), z_{n+1} = 10\left(\frac{1}{2}f_n + \frac{3g_n}{2\sqrt{19}}\right) \quad (13)$$

Thus, (11) and (13) represent the integer solutions to (1).

Illustration 3:

The substitution of the linear transformations

$$z = u + h, x = u - k, u \neq k \neq 0, h \neq k \neq 0 \quad (14)$$

in (1) leads to

$$y^2 = 2(h+k)u + h^2 - k^2 - 10 \quad (15)$$

Remember that h, k are non-zero distinct integers and it is possible to choose them

Such that the R.H.S. of (15) is a perfect square and the value of y is obtained. Substituting the values of u, h, k in (14), the corresponding values of x, z are found. A few numerical examples are exhibited in Table 1: below:

Table 1: Examples

h	k	x	y	z
3	2	$10s^2 - 10k + 1$	$10s - 5$	$10s^2 - 10k + 6$
5	2	$14s^2 - 18s + 3$ $14s^2 - 10s - 1$	$14s - 9$ $14s - 5$	$14s^2 - 18s + 10$ $14s^2 - 10s + 6$

Illustration 4:

The substitution of the linear transformations

$$y = kx, k \neq 1 \quad (16)$$

in (1) leads to the positive pell equation

$$z^2 = (k^2 + 1)x^2 + 10 \quad (17)$$

for which the integer solutions exist when k takes particular values.

Example :3

Considering the value of k to be 3 in (16), it gives the positive pell equation

$$z^2 = 10x^2 + 10 \quad (18)$$

After some algebra, the corresponding integer solutions to (18) are given by

$$x_{n+1} = \frac{3}{2}f_n + \frac{\sqrt{10}}{2}g_n \quad (19)$$

$$z_{n+1} = \frac{1}{2}(10f_n + 3\sqrt{10}g_n) \quad (20)$$

where

$$f_n = (19 + 6\sqrt{10})^{n+1} + (19 - 6\sqrt{10})^{n+1}, g_n = (19 + 6\sqrt{10})^{n+1} - (19 - 6\sqrt{10})^{n+1}$$

In view of (16), we have

$$y_{n+1} = \frac{3}{2}(3f_n + \sqrt{10}g_n) \quad (21)$$

Thus, (19), (20) and (21) represent the integer solutions to (1).

Example :4

Considering the value of k to be 5 in (16), it gives the positive pell equation

$$z^2 = 26x^2 + 10$$

After some algebra, the corresponding integer solutions to (1) are given by

$$x_{n+1} = \frac{1}{2}f_n + \frac{3}{\sqrt{26}}g_n$$

$$z_{n+1} = \frac{1}{2}(6f_n + \sqrt{26}g_n)$$

$$y_{n+1} = 5\left(\frac{1}{2}f_n + \frac{3}{\sqrt{26}}g_n\right)$$

where

$$f_n = (51 + 10\sqrt{26})^{n+1} + (51 - 10\sqrt{26})^{n+1}, g_n = (51 + 10\sqrt{26})^{n+1} - (51 - 10\sqrt{26})^{n+1}$$

Illustration 5:

The substitution of the linear transformation

$$x = y + h, h \neq 0 \quad (22)$$

in (1) leads to the negative pell equation

$$(2y + h)^2 = 2z^2 - (h^2 + 20) \quad (23)$$

for which the integer solutions exist when h takes particular values.

Example :5

Considering the value of h to be 4 in (23), it gives the negative pell equation

$$(2y + 4)^2 = 2z^2 - 36 \quad (24)$$

In this case, the corresponding integer solutions to (24) are given by

$$y_{n+1} = \frac{1}{2} \left(3f_n + \frac{6}{\sqrt{2}} g_n - 4 \right) \quad (25)$$

$$z_{n+1} = \frac{1}{2} (6f_n + 3\sqrt{2} g_n) \quad (26)$$

where

$$f_n = (3 + 2\sqrt{2})^{n+1} + (3 - 2\sqrt{2})^{n+1}, g_n = (3 + 2\sqrt{2})^{n+1} - (3 - 2\sqrt{2})^{n+1}$$

Using (25) in (22), one obtains that

$$x_{n+1} = \frac{1}{2} \left(3f_n + \frac{6}{\sqrt{2}} g_n + 4 \right) \quad (27)$$

Thus, (25), (26) and (27) give the integer solutions to (1).

Example :6

Considering the value of h to be 6 in (23), it gives the negative pell equation

$$(2y + 6)^2 = 2z^2 - 56 \quad (28)$$

In this case, the corresponding integer solutions to (28) are given by

$$y_{n+1} = \frac{1}{2} (2f_n + 3\sqrt{2} g_n - 6) \quad (29)$$

$$z_{n+1} = (3f_n + \sqrt{2} g_n) \quad (30)$$

where

$$f_n = (3 + 2\sqrt{2})^{n+1} + (3 - 2\sqrt{2})^{n+1}, g_n = (3 + 2\sqrt{2})^{n+1} - (3 - 2\sqrt{2})^{n+1}$$

Using (29) in (22), one obtains that

$$x_{n+1} = \frac{1}{2} (2f_n + 3\sqrt{2} g_n + 6) \quad (31)$$

Thus, (29), (30) and (31) give the integer solutions to (1).

3. CONCLUSION

In this paper, an attempt has been made to obtain different sets of non-zero distinct integer solutions to the ternary quadratic diophantine equation $x^2 + y^2 = z^2 - 10$. As diophantine equations are rich in variety, the readers of this paper may search for choices of the integer solutions to the other forms of ternary quadratic diophantine equations.

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