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A study of graph theoretic properties on clean graph of rings and some of their complements for some rings of prime order

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**Abstract**

Let R be a ring of order ’*p*’ such that ’*p*’ is a prime number and *p ≥* 3. A clean element *x* of *R* is an element that can be written as a sum of an idempotent and a unit of *R*. A ring *R* is clean if all its elements are clean. A clean graph of a ring *Cl*(*R*) is graph whose vertices are of the form (*e, u*) where *e* is an idempotent and *u* is a unit in *R* and two vertices (*e, u*) and (*f, v*) are connected if and only if either *ef* = 0 or *uv* = *vu* = 1.In this paper, we study some graph theoretic properties of *Cl*(*R*) for commutative unitary rings of prime order **Keywords;** Idempotent, Unit, Clean element, Clean graph

# Introduction

A graph is conceptually speaking a collection of points or nodes or vertices and lines connecting those vertices under certain condition(s). An algebraic graph is a graph constructed to visualise an algebraic structure where the elements of the structure serve as vertices of the graph and a particular property of the structure will serve as an edge connecting the concerned vertices (elemnts) see [7] , [14]

The study of algebraic structures using properties of graph has become an interesting research topic over the past few decades. There are many papers which study the graphs associated with rings(Pongthana 2022) Many graphs have been constructucted and studied about about different algebraic structures such as semi- groups, groups rings and fields, the essence of these graphs is to visualise an algebraic structure and study its properties. Some of the most popular graphs among them are Cayley graph of a group, identity graph of groups, commuting graph of groups, power graph of group, subgroup graph of groups, zero divisor graph of a rings, prime graph of a rings, graph of units of a ring, ideal graph of rings and clean graph of rings. see [1, 2, 4, 8, 10, 9, 5, 6, 13, 3, 11, 12] and [15]

The concept of zero divisor graph and that of graph theoretic properties of commutative rings was first introduced by Beck (), where all elements of the ring serve as vertices of the graph. Also, Anderson and Vingston introduced zerodivisor graph of commutative rings where nonzero zerodivisors of a ring serve as vertices of the graph and two vertices *a* and *b* are adjacent iff *ab* = 0.After that, the attention of researchers was then drown to the area of study, and different studies have been conducted with respect to graphs associated to rings. This research will focus on one of these graphs namely clean graph of a ring and study some of its graph theoretic properties. Many researches have recenly been conducted with respect to the clean graph of a ring. Habib et al [10] introduced the graph, he constructed and study subgraphs of clean graphs for only some rings of even order ( *Cl*2(*R*)) and determined its clique number , Matching number, independent number and domination number,he also study connectedness of this subgraphs but does not consider the general structure of the graph which gives a room to further determine these properties and even more uninvestigated properties of the structure of other rings with different order such as rings of odd or prime order *Cl*(*R*).

A ring *R* is also an algebraic structure but with two binary operations i.e (*R,* +*,* ·) where *R* is a nonempty set and the two binary operations are namely addition and multiplication respectively such that

Throughout this paper, *R* will be a associative unital ring. The ring of integers and the ring of integers modulo *n* will be respectively denoted by Z and Z*n*. An element *a* ∈ *R* is said to be a unit *u* in *R* if ∃*x* ∈ *R* such that *a* · *x* = 1, where 1 is called the unity of *R*, an idempotent (*e*) in *R* if *a*2 = *a*. and a clean if *a* = *e* + *u*. here, we adopt the convention in some context and represent a clean element *a* of *R* in the form *a* = (*e, u*). A ring is commutattive/ abelian if ∀*x, y* ∈ *R, x.y* = *y.x*

As idempotent elements and units play a major role in the clean graph, we are interested in the number of

idempotent elements in *Zpm* (Pongthana, 2022)

A graph of a clean ring also called clean graph of a ring is a graph whose vertices are clean elements of the ring



Figure 1: Clean graph of the ring Z5

*R* usually in the form (*e, u*) where *e* and *u* are respectively an idempotent and a unit of R, two vertices (*e, u*) and (*f, v*) are connected if and only if either *ef* = 0 or *uv* = *vu* = 1

# Results

* 1. **Some Properties of** *Cl*(*R*)

In this section we determined sizes of *Cl*(*R*) and sizes of some of its complement types.

**Observarions**

Let R be a finite commutative ring with unity of order *n* with *n* a prime number greater than or equal to 5 and let *G*(*U* (*R*)) be graph of units of *R*. Then

* + 1. *cl*(*R*) is a simple graph with no loops or multiple edges
		2. *cl*(*R*) is connected graph
		3. The order of *cl*(*R*) is the total number of vertices in it and is given by |*V* (*cl*(*R*))| = |*id*| × |*U* (*R*)|
		4. *cl*(*R*) is never null graph since *R* is nonempty set.

**Proporsition 2.1.** *Let R be a finite commutative ring with unity of order n with n a prime number, let cl*(*R*) *be clean graph of R, then the degree of the vertices of cl*(*R*) *partitioned V* (*cl*(*R*)) *in to three subsets namely S*1*, S*2 *and S*3 *where S*1 = {(0*, x*)|*x* ∈ *U* (*R*)} = {(0*, x*1)*,* (0*, x*2)*,* · · · *,* (0*, xp*−1)} ⇒ |*S*1| = (*p* − 1)

*S*2 = {(1*, x*)|*x* ∈ *U* ”(*R*)} = {(1*, x*1)*,* (1*, x*2)*,* · · · (1*, xp*−3)} ⇒ |*S*2| = (*p* − 3)

*S*3 = {(1*, x*)|*x* ∈ *U* (*R*)} = {(1*, x*1)*,* (1*, x*2)*,* · · · *,* (1*, x*|*id*∗(*R*)|)} ⇒ |*S*3| = |*id* (*R*)|

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**Proporsition 2.2.** *Let R be a finite commutative ring with unity of order n with n a prime number, let cl*(*R*)

*be clean graph of R, then the degree of each vertex vi* ∈ *Si is given by;*

*deg*(*v*) = { ( 2*p* − 3) *ifv* ∈ *S*1

*(p-1) if v* ∈ *S*3

*p if v* ∈ *S*2

*Proof.* Let R be a finite commutative ring with unity of order *n* with *n* a prime number, let *cl*(*R*) be clean graph of *R*,

Since order of *R* is prime, by lemma 3 |*id*(*R*)| = 2 and there are *p* − 1 units in *R* (because all nonzero elements of *R* are units in *R*).Hence by lemma 2 and defination of *cl*(*R*), the order of *V* (*cl*(*R*)) follows.

Based on the information above, observe that by defination of *cl*(*R*), the element 0 ∈ *id*(*R*) will dorm *p* − 1 vertices with all elements of *U* (*R*) so also the element 1 ∈ *id*(*R*) will form same number of vertices with elements of *U* (*R*), this will make a sum of 2(*p* − 1) = 2*p* − 2 vertices which is exactly |*V* (*cl*(*R*))|

Now, each of the first *p* − 1 vertices of the form (0*, x*) (i.e elements of *S*1)will be adjacent to all of the vertices in *V* (*cl*(*R*)) except itself because 0*.e* = 0∀*e* ∈ *id*(*R*) making its degree to be 2*p* − 2 − 1 = 2*p* − 3 and that is the degree for each of the vertices of *S*1.

Secondly, observe that |*S*2| is given by proposition 2.1 and each of the vertices in *S*2 is adjacent to all of the vertices in *S*1 and is also adjacent to one more vertex (1*, xi*) of *S*2 such that *xi* is inverse of *x* in *U* ′”(*R*) thus

making *deg*((1*, x*)) in *S*2 to be |*S*1| + 1 = (*p* − 1) + 1 = *p* and that is the degree of each vertex in *S*2.

Lastly,the vertices in *S*3 are of the form (1*, x*)|*x* ∈ *U* ′(*R*). This implies that the unit part of each vertex in*S*3 is self invertible in *R* and therefore each of them will only be adjacent to the vertices of *S*1. Thus, *deg*(*v*)|*v* ∈ *S*3 = |*S*1| = *p* − 1.

The proof is complete.

**Remark 2.1.** *i From proposition 4.4 and proposition 2.5, it is observed that the order of the underlined ring is equal to the degree of a veretex in S*2 *that is* |*R*| = *deg*(*v*|*v* ∈ *S*2)

1. *We use the following for degree of vertex in S*1*, S*2 *and S*3 *deg*(*v*|*v* ∈ *S*1) = *d*1

*deg*(*v*|*v* ∈ *S*2) = *d*2

*and deg*(*v*|*v* ∈ *S*3) = *d*3

1. |*S*3| = (*p* − 1) = *d*3

*For order of S*1*, S*2*, S*3 *and values of d*1*, d*2 *and d*3 *see proposition 2.1 and 2.2*

**Lemma 2.1.** *Let Cl*(*R*) *be clean graph of a commutative ring with unity that has prime order, then*

1. *sum of degrees of two adjacent vertices in S*1 *is SD*1 = 2*d*1
2. *sum of degrees of two adjacent vertices between S*1 *and S*2 *is SD*1*,*2 = 3*d*3
3. *sum of degrees of two adjacent vertices between S*1 *and S*3 *is SD*1*,*3 = 3*d*2 − 4
4. *sum of degrees of two adjacent vertices in S*2 *is SD*2 = 2*d*2

*Proof.* i Since from proposition 2.5 all vertices in *S*1 have equal degrees, we have from thesame proposition the sum of degrees of a pair of adjacent vertices in *S*1 as

*SD*1 = (2*p* − 3) + (2*p* − 3) = 2(2*p* − 3)

and by (ii) of remark 1 we have

*SD*1 = 2*d*1

1. From proposition 2.5, the degree of a vertex in 1 is 2*p* − 3 and degree of a vertex in *S*2 is *p*. Therefore the sum of degrees of a pair of adjacent vertices between *S*1 and *S*2 is

*SD*1*,*2 = (2*p* − 3) + *p* = 3*p* − 3 = 3(*p* − 1)

and by (ii) of remark 1 we have

*SD*1*,*2 = 3*d*3

1. From proposition 2.5, the degree of a vertex in *S*1 and *S*3 are (2*p* − 3) and (*p* − 1) respectively. therefore

*SD*1*,*3 = (2*p* − 3) + (*p* − 1) = 3*p* − 4

and by (ii) of remark 1 we have

*SD*1*,*3 = 3*d*2 − 4

1. From proposition 2.5, the degree of a vertex *S*2 is *p*. therefore

*SD*2 = *p* + *p* = 2*p*

and by (ii) of remark 1 we have

The proof is complete

*SD*2 = 2*d*2

**Lemma 2.2.** *Let Cl*(*R*) *be clean graph of a commutative ring with unity that has prime order, then*

1. *Product of degrees of two adjacent vertices in S*1 *is PD*1 = *d*2

1

1. *Product of degrees of two adjacent vertices between S*1 *and S*2 *is PD*1*.*2 = *d*1*d*2
2. *Product of degrees of two adjacent vertices between S*1 *and S*3 *is PD*1*,*3 = *d*1*d*3
3. *Product of degrees of two adjacent vertices in S*2 *is PD*2 = *d*2

2

*Proof.* i From proposition 2.5, the product of degrees of a pair of adjacent vertices in *S*1 is

*PD*1 = (2*p* − 3)(2*p* − 3) = (2*p* − 3)2

and by (ii) of remark 1 we have

*PD*1 = 2*d*2

1

1. From proposition 2.5, the product of degrees of a pair of adjacent vertices between *S*1 and *S*2 is

*PD*1*,*2 = (2*p* − 3)*p*

and by (ii) of remark 1 we have

*PD*1*,*2 = *d*1*d*2

1. From proposition 2.5, the product of degrees of a pair of adjacent vertices between *S*1 and *S*3 is

*PD*1*,*3 = (2*p* − 3)(*p* − 1)

and by (ii) of remark 1 we have

*PD*1*,*3 = *d*1*d*3

1. From proposition 2.5, the product of degrees of a pair of adjacent vertices in *S*2 is is

*PD*2 = *p* × *p* = *p*2

and by (ii) of remark 1 we have THe proof is complete.

*PD*2 = *d*2

2

**Lemma 2.3.** *Let Cl*(*R*) *be clean graph of a commutative ring with unity that has prime order, then*

1. *Number of edges between vertices of S*1 *is N*1 = *d*3 (*d*2 −2)

2

1. *Number of edges between vertices of S*1 *and vertices of S*2 *is N*1*,*2 = *d*3(*d*2 − 3)
2. *Number of edges between vertices of S*1 *and vertices of S*3 *is N*1*,*3 = 2*d*3
3. *Number of edges between vertices of S*2 *only is N*2 = *d*2 −3

2

*Proof.* Let *Cl*(*R*) be clean graph of a commutative ring with unity that has prime order, then

1. From proposition 2.5, we have the number of edges between vertices of *S*1 is given by the size of *S*1 that is

and by (ii) of remark 1 we have

*N*1 =

*N*1

(*p* − 1)(*p* − 2)

2

= *d*1(*d*2 − 2)

2

1. From proposition 2.4 and 2.5, the number of edges between vertices of *S*1 and vertices of *S*2 is given by

*N*1*,*2 = (*p* − 1)(*p* − 3)

because all the (*p* − 1) vertices of *S*1 are adjacent to all the (*p* − 3) vertices of *S*2. Therefore by (ii) of remark 1, we have

*N*1*,*3 = *d*3(*d*2 − 2)

1. From proposition 2.4 and 2.5, the number of edges between vertices of *S*1 and vertices of *S*3 is given by

*N*1*,*3 = (*p* − 1)2

because all the (*p* − 1) vertices of *S*1 are adjacent to all the 2 or |*id*(*R*)| vertices of *S*3. Therefore by (ii) of remark 1, we have

*N*1*,*3 = 2*d*3

1. From proposition 2.4 and 2.5, the number of edges between vertices of *S*2 is given by

*p* − 3

*N*2 = 2

because only half of vertices of *S*2 are adjacents. Therefore by (ii) of remark 1, we have

*p* − 3

*N*2 = 2

**Lemma 2.4.** *Let Cl*(*R*) *be clean graph of a commutative ring with unity that has prime order, then*

1. *The number of pairs of vertices at distance 1 is NP*1 = 1 [*d*3(3*d*2 − 4) + (*d*2 − 3)]

2

1. *The number of pairs of vertices at distance 2 is NP*2 = 5*d*2 −13

2

**Remark 2.2.** *Number of vertices at distance 1 is equal to the size of the clean graph of R. That is*

*NP*1 = |*Cl*(*R*)|

**Proporsition 2.3.** *Let R be a finite commutative ring with unity of order n with n a prime number, let Cl*(*R*)

*be clean graph of R, then the size of Cl*(*R*) *is*

|*E*(*Cl*(*R*))| = 1 (3*p*2 − 6*p* + 1) (1)

2

*Proof.* Let R be a finite commutative ring with unity of order *n* with *n* a prime number, let *Cl*(*R*) be clean graph of *R*, since proposition 2.5 tells us that the degrees of vertices of *Cl*(*R*) partitioned *V* (*Cl*(*R*)) in to *S*1*, S*2 and *S*3 where all vertices of *Si, i* ∈ {1*,* 2*,* 3} have equal degree, we have the total number of vertices of *Cl*(*R*) given by

|*S*1| + |*S*2| + |*S*3|

and the total degrees of vertices of *Cl*(*R*) given by

Σ

*v*∈*V* (*Cl*(*R*)) *deg*(*v*) = *S*1|(*deg*(*vi*)|*vi* ∈ *S*1) + |*S*2|(*deg*(*vj*)|*vj* ∈ *S*2) + |*S*3|(*deg*(*vk*)|*vk* ∈ *S*3)

= Σ*p*−1(2*p* − 3) + Σ*p*−3 *p* + 2(*p* − 1

1 1

= (*p* − 1)(2*p* − 3) + (*p* − 3)*p* + 2(*p* − 1)

= (*p* − 1)[(2*p* − 3) + 2] + *p*(*p* − 3)

= (*p* − 1)(2*p* − 1) + *p*(*p* − 3)

= 3*p*2 − 6*p* + 1 Now, using **lemma 1** we have 2—E(Cl(R))—=3p2 − 6*p* + 1

⇒ |*E*(*Cl*(*R*))| = 1 (3*p*2 − 6*p* + 1)

2

The proof is complete

**Proporsition 2.4.** *Let R be a finite commutative ring with unity of order n with n a prime number, let cl*(*R*)

*and cl*′(*R*) *be clean graph of R and its complement respectively, then the size of cl*′(*R*) *is*

|*E*(*Cl*′(*R*))| = 1 (*p*2 − 4*p* + 5) (2)

2

*Proof.* Let R be a finite commutative ring with unity of order *n* with *n* a prime number, let *Cl*(*R*) and *Cl*′(*R*) be clean graph of *R* and its complement respectively, then, since |*V* (*cl*(*R*))| = 2*p* − 2, by defination ...(of complete graph on n vertices *Kn*), each vertex has degree *n* − 1. in this case *n* = 2*p* − 2 which implies that in *K*|*V* (*Cl*(*R*))| = *K*2*p*−2, each vertex has degree 2*p* − 3

Now for *Cl*(*R*), consider vertices in the set *S*1 where degree of each is 2*p* − 3 (they contain thier maximum

degrees in the graph), by proposition 2.3 *S*1 is a clique of *cl*(*R*) and will therefore be an independent set in

*Cl*′(*R*). Thus, in *Cl*′(*R*) *deg*(*v*)|*v* ∈ *S*1 is 0

Secondly, in *cl*(*R*)*, deg*(*u*)|*u* ∈ *S*2 is *p* which implies that *deg*(*u*)|*u* ∈ *S*2 will be (2*p* − 3) − *p* = *p* − 3 in *cl*′(*R*)

Lastly, the vertices in *S*3 are 2 and *deg*(*v*)|*v* ∈ *S*3 = *p* − 1 in *Cl*(*R*) which implies that *deg*(*v*)|*v* ∈ *S*3 = (2*p* − 3) − (*p* − 1) = *p* − 2 in *Cl*′(*R*)

We therefore have the sum of the degrees of vertices of *cl*′(*R*) as follows; —E(Cl’(R))—=—S1|(*deg*(*v*)|*v* ∈

*S*1) + |*S*2|(*deg*(*v*)|*v* ∈ *S*2) + |*S*3|(*deg*(*v*)|*v* ∈ *S*3)

= Σ*p*−1 0 + Σ*p*−3(*p* − 3) + 2(*p* − 2)

1 1

= (*p* − 3)2 + 2(*p* − 2) = *p*2 − 4*p* + 5

⇒ |*E*(*Cl*′(*R*))| = 1 (*p*2 − 4*p* + 5)

2

Or equivalently, E(Cl’(R))—=—E(K|*V* (*Cl*(*R*))|| − |*E*(*Cl*(*R*))|

= |*E*(*K*2*p*−2)| − 1 (3*p*2 − 6*p* + 1)

2

= (2*p*−2)(2*p*−3) − 1 (3*p*2 − 6*p* + 1)

2 2

= 4*p*2−10*p*+6−3*p*2+6*p*−1

= 1 (*p*2

2

2

— 4*p* + 5)

The proof is complete.

* 1. **The subgraph** *Cl*2(*R*))

According to (...) *Cl*2(*R*) is a subgraph of *Cl*(*R*) induced by the set *id*∗ of nonzero idempotent of the ring *R*. Baesd on the nature of the order of the ring we are considering in this work, this subgraph has *p* − 1 vertices because the only nonzero idempoteent of this ring is the element 1. We therefore explore properties of this type of the induced subgraph.

*Cl*2(*R*) is a simple and disconnected graph with *p*+1 components

2

**Proporsition 2.5.** *Let R be a finite commutative ring with unity of order n with n a prime number, let Cl*(*R*)

*be clean graph of R, then the subgraph Cl*2(*R*) *of Cl*(*R*) *is a disconnected graph with p*+1 *components.*

2

**Proporsition 2.6.** *Let R be a finite commutative ring with unity of order n with n a prime number, let Cl*(*R*)

*be clean graph of R, then two vertices* (*e, u*) *and* (*f, v*) *are adjacents in Cl*2(*R*) *if and only if uv* = *vu* = 1

**Proporsition 2.7.** *Let R be a finite commutative ring with unity of order n with n a prime number, let Cl*2(*R*) *be be induced subgraph of the clean graph Cl*(*R*)*, then the degrees of the vertices of Cl*2(*R*) *partitioned V* (*Cl*(*R*)) *into two subsets V*1 *and V*2 *where all the vertices of a given subset contain equal degrees and* |*V*1| = 2 *and*

|*V*2| = (*p* − 3)*. Furthermore, for any vertex v, deg*(*v*)|*v* ∈ *V*1 = 0 *and deg*(*v*)|*v* ∈ *V*2 = 1*.*

*Proof.* Let R be a finite commutative ring with unity of order *n* with *n* a prime number, let *Cl*2(*R*) be be induced subgraph of the clean graph *Cl*(*R*), then, clearly *Cl*2(*R*) has *p* − 1 vertices because |*id*∗| = 1 and |*U* (*R*)| = *p* − 1. Secondly, since *p* is prime, *p* − 3 out of*p* vertices of *Cl*(*R*) will satisfy the condition in proposition 2.8 (i.e excluding vertices of the form (1*, u*)|*u* ∈ *U* ′(*R*))and using the fact that 0 ∈*/ U* (*R*), thus *V*2 = *S*1 and since |*V*2| is even, by proposition 2.8 and lemma 1 *p*−3 pairs of adjacent vertices will be formed in *cl*2(*R*) hence for each vertex *v* ∈ *Cl*2(*R*)*, deg*(*v*) = 1 amd therefore |*V*2| = *p*−*,* 3. Lastly these excluded vertices are (1*,* 1) and (1*, p* − 1) and are isolated in *Cl*2(*R*), hence *d*((1*,* 1)) = *deg*((1*, p* − 1)) = 0. This forms the first partion of *V* i,e *V*1 and it is obvious that |*V*1| = |{(1*,* 1)*,* (1*, p* − 1)}| = 2

2

The proof is complete.

**Proporsition 2.8.** *Let R be a commutative ring with inity If Cl*(*R*) *is a clean graph of R, then the size of*

*Cl*2(*R*) *is given by*

1

|*E*(*Cl*2(*R*))| = 2 (*p* − 3) (3)

*Proof.* From proposition 2.4, elements of *S*1 are exacly the vertices of *Cl*2(*R*) i.e *p* − 1 vertices, out of these *p* − 1 vertices, (1*,* 1) and *p* − 1 are self inverses and by proposition 2.9, they are not adjacent to any other vertex in rhe graph, thus the remaining *p* − 3 will by proposition 2.2 form *p*−3 pairs of adjacent vertices.

3

Hence the proposition

**Proporsition 2.9.** *Let R be a commutative ring with inity If Cl*(*R*) *is a clean graph of R, then the size of the complement of Cl*2(*R*) *is given by*

|*E*(*Cl*′ (*R*))| = 1 (*p*2 − 4*p* + 5) (4)

2 2

*Proof.* Let R be a commutative ring with inity If *Cl*(*R*) is a clean graph of *R*,then Since *Cl*2(*R*) has *p* − 1 vertices, so by lemma ()a complete graph on *p* − 1 vertices is given by

E(K*p*−1)| = (*p*−1)(*p*−2)

2

Therefore, the size of the complement of *Cl*2(*R*) is given by

—E(Cl2(*R*))| = |*E*(*Kp*−1)| − |*E*(*cl*2(*R*))|

= (*p*−1)(*p*−2) − 1 (*p* − 3)

2 2

(*p*−1)(*p*−2)−(*p*−3)

=

2

= *p*2 −4*p*+5 The proof is complete,

2

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